

Scattering of spin-polarized electron in an Aharonov–Bohm potential

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The scattering of spin-polarized electrons in an Aharonov–Bohm vector potential is considered. We solve the Pauli equation in 3+1 dimensions taking into account explicitly the interaction between the three-dimensional spin magnetic moment of electron and magnetic field. Expressions for the scattering amplitude and the cross section are obtained for spin-polarized electron scattered off a flux tube of small radius. It is also shown that bound electron states cannot occur in this quantum system. The scattering problem for the model of a flux tube of zero radius in the Born approximation is briefly discussed.

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I. INTRODUCTION

The quantum Aharonov–Bohm (AB) effect, predicted by Aharonov and Bohm [1], has been analyzed in various physical situations in numerous works (see e.g., Ref. [2]). It occurs when an electron travels in a certain configuration of a vector potential A_μ in which the corresponding magnetic flux is restricted to a finite-radius (R) tube topologically equivalent to a cylinder. When an electron travels in an AB potential the electron wave function acquires a (topological) phase which could have physical effects on the behavior of the electron, such as the interference pattern in the two-slit experiment. The AB vector potential can produce observable effects because the relative (gauge invariant) phase of the electron wave function, correlated with a nonvanishing gauge vector potential in the domain where the magnetic field vanishes, depends on the magnetic flux [3].

When the external field configuration has the cylindrical symmetry, a natural assumption is that the relevant quantum mechanical system is invariant along the symmetry (z) axis and the system then becomes essentially two-dimensional in the xy plane. So, the models applied to describe AB effect can usually be reduced to the (2+1)-dimensional ones.

The results of Ref. [1] for nonrelativistic case modified by using the Dirac equation in 2+1 dimensions were applied to other problems. Solutions of the two-component Dirac equation in the AB potential were first discussed by Alford and Wilczek in Ref. [4] in a study of the interaction of cosmic strings with matter. Relativistic quantum AB effect was studied in Ref. [5] for the free and bound fermion states by means of exact analytic solutions of the Dirac equation in 2+1 dimensions for a combination of an AB potential and the Lorentz three-vector and scalar Coulomb potentials.

Note that the usual four-component Dirac equation in 2+1 dimensions (in the absence of z coordinate) decouples into two uncoupled two-component Dirac equations for spin projection $s = +1$ and $s = -1$. Thus, the two-component Dirac equation describes the planar motion of relativistic electron having only one projection of three-dimensional spin vector. The upper (“large”) and lower (“small”) components of the two-component wave function are interpreted in terms of positive- and negative-energy solutions of the Dirac equation in 2+1 dimensions.

The scattering of spin-polarized fermions in an Aharonov–Bohm potential was first considered in 2+1 dimensions by Hagen in [6]. The particle spin in [6] is artificially introduced into the two-component Dirac equation as a new parameter (see, also, [7]). The term including this new parameter appears in the form of an additional delta-function interaction of spin with magnetic field in the Dirac equation. Solutions of the this Dirac equation were then interpreted for the case of 3+1 dimensions. The magnetic field strength H was taken to have the form

$$H = -\frac{a}{R}\delta(r - R), \quad a, R : \text{constants}, \quad (1)$$

and the vector potential in the Coulomb gauge is specified as

$$A^0 = 0, \quad A_r = 0, \quad A_\varphi = \frac{a}{r}, \quad r > R; \quad \mathbf{A} = \mathbf{0}, \quad r < R.$$

Even if the limit $R \rightarrow 0$ is taken at the end of the calculation, it is seen that the above magnetic configuration does not quite correspond to the real case because the magnetic field is equal to zero inside

the tube. So the question still remain as to how the interaction of the electron spin with the magnetic field of flux tube can modify the known AB phenomenon.

In this paper we would like to consider the effect of spin in the scattering process of a spin-polarized electron off an AB flux tube with a small but finite radius. In order to take account of the interaction between the three-dimensional spin magnetic moment of electron and magnetic field we shall use solutions of the Pauli equation in 3+1 dimensions, which contains the corresponding spin term explicitly.

This paper is organized as follows. In Sec. II we find the scattering states of electrons in an AB potential, and give a semiclassical argument for the scattering of spin-polarized electrons in an AB potential. In Sec. III we determine the scattering states of electrons for the realistic case when the magnetic field is concentrated inside the cylindrical tube of a small radius, and the scattering cross section for spin-polarized electrons. Possibility of the existence of bound states in an AB potential is briefly discussed in Sect. IV. In Sec. V we briefly consider the scattering problem for the model of a flux tube of zero radius in the Born approximation. Sect. VI concludes the paper. In the Appendix, we briefly discuss the scattering of a spin-polarized Dirac electron in an AB potential in 2+1 dimensions.

II. SCATTERING OF A SPIN-POLARIZED ELECTRON OFF AN AB FLUX TUBE OF ZERO RADIUS

The problem of scattering of a spin-polarized electron described by a Pauli equation off an AB flux tube of zero radius was studied in [8]. For completeness we give the main results here.

A. Scattering states

Consider an electron of mass m and charge $e < 0$ in an AB potential, which is specified in Cartesian or cylindrical coordinates as

$$\begin{aligned} A^0 &= 0, \quad A_x = -\frac{By}{r^2}, \quad A_y = \frac{Bx}{r^2}, \quad A_z = 0; \\ A^0 &= 0, \quad A_r = 0, \quad A_\varphi = \frac{B}{r}, \quad A_z = 0, \quad B = \frac{\Phi}{2\pi}, \\ r &= \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x). \end{aligned} \quad (2)$$

This potential describes the magnetic field of an infinitely thin solenoid with a finite magnetic flux Φ in the z direction. The magnetic field \mathbf{H} is restricted to a flux tube of zero radius

$$\mathbf{H} = (0, 0, H) = \nabla \times \mathbf{A} = B\pi\delta(\mathbf{r}). \quad (3)$$

To take into account of spin, we consider in this paper the Pauli equation of a spinor $\Psi(t, \mathbf{r}, z)$

$$i\hbar \frac{\partial}{\partial t} \psi(t, \mathbf{r}, z) = \mathcal{H} \psi(t, \mathbf{r}, z), \quad \mathbf{r} = (x, y), \quad (4)$$

where the Hamiltonian \mathcal{H} is

$$\mathcal{H} = \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} + \frac{eBy}{cr^2} \right)^2 + \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial y} - \frac{eBx}{cr^2} \right)^2 - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + M\sigma_3 H. \quad (5)$$

Here $M = |e|\hbar/2mc$ is the Bohr magneton and σ_3 is the Pauli spin matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The last term in (5) describes the interaction of the electron spin with a magnetic field.

One seeks solutions of the Pauli equation in the form

$$\begin{aligned} \Psi(t, r, \varphi, z) &= \exp(-iEt/\hbar + ik_z z/\hbar) f(r, \varphi) \psi \\ &\equiv \exp(-iEt/\hbar + ik_z z/\hbar) \sum_{l=-\infty}^{\infty} F_l(r) \exp(il\varphi) \psi, \end{aligned} \quad (6)$$

where E is the electron energy, k_z is the wave number in the z -direction, l is an integer, and ψ is a constant two-spinor. The function $F_l(r)$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - k_z^2 - \frac{l^2}{r^2} - \frac{2|e|Bl}{c\hbar r^2} - \frac{e^2 B^2}{c^2 \hbar^2 r^2} + \frac{2Em}{\hbar^2} - \frac{|e|Bs}{c\hbar} \frac{\delta(r)}{r} \right) F_l(r) = 0. \quad (7)$$

Here the number $s = \pm 1$ characterizes the electron spin projection on the z axis. Eq. (7) is satisfied by the function $F_l(r)$ everywhere except for the point $\mathbf{r} = 0$. Hence, in the range $r > 0$ the linearly-independent solutions for $F_l(r)$ are expressed as

$$F_l(r) = a_l J_\nu(k_\perp r) + b_l J_{-\nu}(k_\perp r), \quad (8)$$

where $J_\nu(k_\perp r)$ and $J_{-\nu}(k_\perp r)$ are the usual Bessel functions, and

$$\nu = |l + \gamma| \neq 0, \quad \gamma = |e|B/\hbar c > 0, \quad k_\perp = \sqrt{2mE/\hbar^2 - k_z^2}, \quad (9)$$

a_l and b_l are constants. For $\nu > 0$ the regular solution is

$$F_l(r) = J_\nu(k_\perp r). \quad (10)$$

For $\nu = 0$ the linearly-independent solutions for $F_l(r)$ are the Bessel ($J_0(k_\perp r)$) and Neumann ($N_0(k_\perp r)$) functions. Note that when ν is an integer the magnetic flux is quantized.

The electron wave function (6) with $E > 0$ belongs to the continuous spectrum and describe the states of scattering. For $B = 0$ one recovers the free electron solutions from Eq.(10). One sees from Eqs. (5) and (7) that the additional singular “potential” acts only at the point $\mathbf{r} = 0$ where the regular solution $J_\nu(k_\perp r)$ goes to zero, so we conclude that the regular solutions are well valid in the range $r > 0$ if only new possible (bound) states do not occur.

One can define the scattering amplitude in the conventional manner. Assuming that the electron wave incidents from the left along the x axis. Hence, $k_z = 0$ and k_\perp reduces to $k \equiv \sqrt{2mE/\hbar}$. The incident wave function is $\psi = e^{ikx}$. For this case φ is the scattering angle measured from the positive x -axis. As $r \rightarrow \infty$, the electron wave function have the asymptotic form

$$\psi_p(r, \varphi) = e^{ikx + i(|e|B/\hbar c)\varphi} + \frac{f(\varphi)}{\sqrt{r}} e^{ikr}. \quad (11)$$

Here $f(\varphi)$ is the scattering amplitude. The scattering amplitude is proportional to $S_l - 1 \equiv e^{2i\delta_l} - 1$, where $\delta_l = (\nu - l)\pi = \pi\gamma$ are the partial phase shifts. They depend upon only the total magnetic flux. Let us write $\gamma = n + \mu$, where n is an integer and $0 \leq \mu < 1$. The scattering amplitude is then found to be given by the AB formula

$$f_{AB}(\varphi) = \sqrt{\frac{i}{2\pi k}} \frac{e^{-i\varphi(n-1/2)} \sin(\pi\gamma)}{\sin(\varphi/2)}. \quad (12)$$

This formula has also been obtained by Alford and Wilczek in Ref. [4] (see, also [5]) by solving the Dirac equation in 2+1 dimensions in an AB potential. Formula (12) is the scattering amplitude of unpolarized electrons, and is apparently independent of the electron spin. Thus the corresponding cross section is the same as that given by the known AB formula

$$\frac{d\sigma}{d\varphi} = |f_{AB}(\varphi)|^2 = \frac{\sin^2(\pi\gamma)}{2\pi k \sin^2 \varphi/2}. \quad (13)$$

B. Inclusion of spin

Let us now discuss how the inclusion of electron spin polarized along the direction of the a unit vector \mathbf{n} may modify the scattering amplitude. In the range $r > 0$ the spinor ψ can be determined from the following equation

$$(\boldsymbol{\sigma} \cdot \mathbf{n})\psi^s = s\psi^s, \quad (14)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, $\mathbf{n} = (\sin\vartheta, 0, \cos\vartheta)$ is the unit vector characterized by the polar (ϑ) and azimuthal ($\phi = 0$) angles with respect to the fixed axes x, y, z , and $s = \pm 1$ is the number which characterizes the electron spin projection on the direction of the unit vector \mathbf{n} . The solution of Eq. (14) is given by

$$\psi^s = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + s(\mathbf{n} \cdot \mathbf{n}_z)} \\ -s\sqrt{1 - s(\mathbf{n} \cdot \mathbf{n}_z)} \end{pmatrix}, \quad (15)$$

where $(\mathbf{n} \cdot \mathbf{n}_z)$ is a scalar product of \mathbf{n} with the unit vector \mathbf{n}_z along the z -axis.

It follows from Eq. (7) that for the scattering of spin-polarized particles only the following part of the potential depending on the l and s

$$\frac{2|e|Bl}{c\hbar r^2} + \frac{|e|Bs}{c\hbar} \frac{\delta(r)}{r} \quad (16)$$

is essential. One sees from Eq. (16) that there arises no an asymmetry in the scattering for any spin projections $s = 0, \pm 1$. This means that the spin projection is conserved and the dependence on the particle spin in the cross section arises only because the propagation direction of electron is changed after the scattering.

The change in the propagation direction of electron is due to the momentum transfer occurring in the scattering process. Consider the case when the spin of the scattered electron is oriented along the direction of the unit vector \mathbf{n} . Then, the constant spinor ψ^s can be rewritten as

$$\psi^1 = \begin{pmatrix} e^{-i\phi/2} \cos(\vartheta/2) \\ e^{i\phi/2} \sin(\vartheta/2) \end{pmatrix} \quad (17)$$

and

$$\psi^{-1} = \begin{pmatrix} -e^{-i\phi/2} \sin(\vartheta/2) \\ e^{i\phi/2} \cos(\vartheta/2) \end{pmatrix}. \quad (18)$$

Putting $\phi = 0$ or $\phi = \pi/2$ in the initial spin function ψ_i^s , we must put $\phi = \varphi$ or $\phi = \varphi + \pi/2$ in the final spin function ψ_f^s . Furthermore, expanding the initial spin-up and down functions over the spin function in the final state as follows

$$\psi_i^1 = A\psi_f^1 + B\psi_f^{-1}, \quad \psi_i^{-1} = C\psi_f^1 + D\psi_f^{-1}, \quad (19)$$

one can easily find that $A = D^* = \cos(\varphi/2) + i \cos\vartheta \sin(\varphi/2)$, $C = B^* = i \sin\vartheta \sin(\varphi/2)$ for both cases. Note that $\cos\vartheta$ and $\sin\vartheta$ can be related to the scalar product $\mathbf{n} \cdot \mathbf{n}_z$ and vector product $\mathbf{n} \times \mathbf{n}_z$ of the initial unit vector \mathbf{n} and the unit vector \mathbf{n}_z , respectively.

Since the spin orientation cannot be changed, the cross section (13) for spin-polarized electrons with the spin along the direction of the unit vector \mathbf{n} must be multiplied by the additional factors $|A|^2$ or $|D|^2$. Thus, for these cases the cross section is described by following equation (see, also, [8])

$$\frac{d\sigma}{d\varphi} = \frac{\sin^2(\pi\gamma)}{2\pi k} \left(\frac{1}{\sin^2\varphi/2} - (\mathbf{n} \times \mathbf{n}_z)^2 \right). \quad (20)$$

Eq. (20), up to the replacement of φ by $\varphi + \pi$, coincides with the expression for the cross section of polarized beams given in [6]. The difference is due to the fact that the scattering angle here is measured from the positive x -axis, whereas in [6] it is measured from the negative x -axis. It is amazing that our result, using the $(3+1)$ -dimensional Pauli equation with the flux tube concentrating at the origin, happens to coincide with that given in [6], in which the particle spin is artificially introduced into the Dirac equation in $2+1$ dimensions as a new parameter and the magnetic field is concentrated only to the surface of a cylinder of radius R , as mentioned in the Introduction (see Eq. (1)).

III. SCATTERING OF A SPIN-POLARIZED ELECTRON OFF AN AB FLUX TUBE OF SMALL RADIUS

We shall now give a first principle derivation of Eq. (20), using the $(3+1)$ -dimensional Pauli equation, for the case in which the electron spin is perpendicular to the AB flux tube of small but finite radius. Eq. (20) for general spin orientation will then be obtained by interpolating the formula for this case and that for the case of spin parallel to the flux tube.

A. Scattering states

In realistic situation the magnetic field is restricted to a flux tube of small radius R , so the vector potential \mathbf{A} in the cylindrical coordinates r, φ, z inside the flux tube can be specified as

$$A^0 = 0, \quad A_r = 0, \quad A_\varphi = \frac{1}{2}Hr, \quad A_z = 0. \quad (21)$$

In the range $r > R$ the vector potential \mathbf{A} is the AB potential

$$A^0 = 0, \quad A_r = 0, \quad A_\varphi = \frac{B}{r}, \quad A_z = 0, \quad (22)$$

where $2B = HR^2$. The magnetic field \mathbf{H} in the flux tube is

$$\mathbf{H} = (0, 0, H) = \nabla \times \mathbf{A} \quad (23)$$

and is equal to zero outside the tube.

The standard approach is to obtain solutions of the Pauli equation in the range $r < R$ and $r > R$, and then to impose the matching condition at $r = R$. As far as the spin projection on the z axis is conserved we can substitute the eigenvalue $s = \pm 1$ for the operator σ_3 in the Hamiltonian (5). After such a substitution the spin dependence of the wave function becomes unessential and Ψ may be treated as an usual function of position (see, e.g. [9]). Seeking solutions of the Pauli equation in fields (21) and (22) in the form (6), we arrive at the differential equation for $F_l(r)$ in the range $r < R$ in the form

$$\left[\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2}{r^2} - k_z^2 \right) + E - \frac{(eHr)^2}{8mc^2} + \frac{\hbar|e|H(l+s)}{2mc} \right] F_l(r) = 0. \quad (24)$$

In the range $r > R$ solution $F_l(r)$ is determined by Eq.(8).

As we are interested in the situation in which the electron is moving in the xy -plane, we shall put $k_z = 0$ in what follows. The positive-energy solution for $F_l(r)$ regular in the range $r < R$ is

$$F_l(r) = A_l e^{-x/2} x^{|l|/2} \Phi(a, d; x), \quad (25)$$

where A_l is a constant, $x = |eH|r^2/2c\hbar \equiv r^2\gamma/R^2$, $\Phi(a, d; x)$ is the confluent hypergeometric function, and

$$a = - \left(\frac{EmR^2}{2\gamma\hbar^2} - \frac{|l| + l + s + 1}{2} \right), \quad \omega = \frac{|eH|}{mc}, \quad d = |l| + 1. \quad (26)$$

We emphasize that a depends on the spin parameter s . This solution is normalized for any l . A second linearly-independent solution $F_l(r)$ in the range $r < R$ (irregular at $r = 0$) for integers d is given by the confluent hypergeometric function $\Psi(a, d; x)$ in the form (see, for example, [10])

$$F_l(r) = B_l e^{-x/2} x^{|l|/2} \Psi(a, d; x), \quad (27)$$

It should be noted that this solution is normalized only when $l = 0$. Solutions (25) and (27) differ from those obtained in [6]. This is because the magnetic field is equal to zero in the range $r < R$ in the model used in [6].

In the range $r > R$ the linearly-independent solutions for $F_l(r)$ are given by

$$F_l(r) = a_l J_\nu(kr) + b_l J_{-\nu}(kr). \quad (28)$$

For $\nu = 0$ the linearly-independent solutions for $F_l(r)$ are the Bessel ($J_0(kr)$) and Neumann ($N_0(kr)$) functions. The functions $F_l(r)$ are just the Fourier coefficients in the expansion of the spatial wave function.

In the range $r < R$, owing to the term proportional to the σ_3 matrix, the constant spinor ψ is determined from the equation

$$(\boldsymbol{\sigma} \cdot \mathbf{n}_z) \psi^s = s \psi^s, \quad (29)$$

where $s = \pm 1$. Hence, the number $s = \pm 1$ characterizes the electron spin projection on the z axis. The solution of Eq. (29) has the form

$$\psi^s = \begin{pmatrix} \psi^1 \\ \psi^{-1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+s \\ 1-s \end{pmatrix}. \quad (30)$$

Thus, for $r < R$ the spatial wave function depends only on the spin projection in (or opposite) the magnetic field. The form (25) implies that we can consider in the range $r < R$ the spatial wave function depending explicitly on the number s , which selects a particular value as the spin projection on the z axis. The number s will explicitly appear in the continuity relations for the spatial wave functions at the point $r = R$. One sees from Eq. (30) that the magnetic field concentrated in the range $r < R$ can have physical effect on the spin state of the electron if the electron spin does not lie exactly along the z axis in the range $r > R$.

B. Scattering cross section for spin-polarized electrons

Now we must match the spatial wave functions and their derivatives at the point $r = R$ using the continuity relations. Results, obviously, depend upon the orientation of electron spin. Indeed, if in the range $r > R$ the electron spin is oriented in a direction of the flux tube then the continuity relations for the spatial wave functions will not depend upon the number s because the electron spin can only be oriented along the z axis inside the flux tube and its projection is conserved. Therefore, when the unit vectors \mathbf{n} and \mathbf{n}_z are parallel, we can put $s = 0$ in the spatial wave functions. But if the electron spin is not oriented in a direction of the unit vector \mathbf{n}_z then the spatial wave functions depend on the number $s = \pm 1$. Hence, the continuity relations can be written as

$$F_l(R - \delta) = F_l(R + \delta), \quad \delta \rightarrow 0, \quad (31)$$

$$\left(\frac{dF_l(r - \delta, s)}{dr} \right)_{r=R, \delta \rightarrow 0} = \left(\frac{dF_l(r + \delta)}{dr} \right)_{r=R, \delta \rightarrow 0}, \quad (32)$$

where $s = \pm 1$.

Using the formula

$$\frac{d}{dx} \Phi(a, d; x) = \frac{a}{d} \Phi(a + 1, d + 1; x) \quad (33)$$

the continuity relations can be written as

$$A_l e^{-(\gamma/2)(\gamma)^{|l|/2}} \Phi(a, d; \gamma) = a_l J_\nu(kR) + b_l J_{-\nu}(kR), \quad (34)$$

$$A_l \frac{\gamma}{R} e^{-(\gamma/2)(\gamma)^{|l|/2}} \left[\left(\frac{|l|}{\gamma} - 1 \right) \Phi(a, d; \gamma) + \frac{2a}{d} \Phi(a + 1, d + 1; \gamma) \right] = \\ a_l \frac{d}{dR} J_\nu(kR) + b_l \frac{d}{dR} J_{-\nu}(kR), \quad (35)$$

The case $\gamma = |e|HR^2/2\hbar c < 1$ is of interest only. Indeed, we can always obtain this magnitude for γ by choosing the appropriate magnitude of H or R . For $\gamma < 1$ we can keep only the leading terms in the expansion of the confluent hypergeometric function $\Phi(a, d; x)$, and the continuity relations for small γ can be written as

$$C_l \left[1 + \left(\frac{a}{d} - \frac{1}{2} \right) \gamma \right] = N_l(kR)^{|l+\gamma|} \left[1 - \frac{kR}{2(|l+\gamma|+1)} + \frac{(kR)^2}{8(|l+\gamma|+1)(|l+\gamma|+2)} \right] \\ + D_l(kR)^{-|l+\gamma|} \left[1 - \frac{kR}{2(-|l+\gamma|+1)} + \frac{(kR)^2}{8(-|l+\gamma|+1)(-|l+\gamma|+2)} \right], \quad (36)$$

$$C_l \left[|l| + \left(\frac{a}{d} - \frac{1}{2} \right) \gamma(|l|+2) \right] = N_l(kR)^{|l+\gamma|} \left[|l+\gamma| - \frac{kR}{2} + \frac{(kR)^2}{4(|l+\gamma|+1)} \right] \\ + D_l(kR)^{-|l+\gamma|} \left[-|l+\gamma| - \frac{kR}{2} - \frac{(kR)^2}{4(|l+\gamma|-1)} \right], \quad (37)$$

where

$$C_l = A_l(\gamma)^{|l|/2}, \quad N_l = \frac{a_l}{2^{|l+\gamma|}\Gamma(|l+\gamma|+1)}, \quad D_l = \frac{b_l}{2^{-|l+\gamma|}\Gamma(-|l+\gamma|+1)} \quad (38)$$

and $\Gamma(z)$ is the gamma function of argument z .

Note that at $R \rightarrow 0$ the situation in which the electron is most probably found near the origin can only occur when the spin-depending potential is attractive, which requires $\gamma s < 0$ and $l = 0$. Putting in Eqs. (36) and (37) $l = 0$, we obtain

$$C_0 = N_0(kR)^{|\gamma|} + D_0(kR)^{-|\gamma|} \quad (39)$$

and

$$C_0 \frac{\gamma s}{|\gamma|} = N_0(kR)^{|\gamma|} + D_0(kR)^{-|\gamma|}, \quad (40)$$

from which one easily finds

$$N_0(kR)^{|\gamma|} = \frac{1}{2} \left(1 + \frac{s\gamma}{|\gamma|} \right) C_0, \quad (41)$$

$$D_0(kR)^{-|\gamma|} = \frac{1}{2} \left(1 - \frac{s\gamma}{|\gamma|} \right) C_0. \quad (42)$$

For $\gamma > 0$, $s = -1$, we have $N_0 = 0$, which implies that the solution $F_0(r) \rightarrow J_{-|\gamma|}(kr)$ in the range $r > R$.

It should be emphasized again that the s term occurs in the continuity relations only when the electron spin is not oriented in the z direction for $r > R$. It follows from Eqs. (34) and (36) that at $R \rightarrow 0$ all the coefficients D_l vanish when the electron spin lies in the z axis and only the regular functions $J_\nu(kr)$ are present in the expansion of the spatial electron wave function in the range $r > R$. Therefore, in case the electron spin is directed along the z axis in the range $r > R$ the spatial electron wave function $\psi(r, \varphi)$ is given by

$$\psi(r, \varphi) = \sum_{l=-\infty}^{\infty} N_l J_\nu(kr) e^{il\varphi}, \quad (43)$$

where

$$N_l = e^{-i(\pi/2)|l+\gamma|}. \quad (44)$$

From this it is easily checked that the scattering amplitude is given by the AB formula (12).

Now we turn to the situation where the electron spin is perpendicular to the flux tube. We have derived that for $1 > \gamma > 0$, $s = -1$ the coefficient N_0 is equal to zero. This implies that the $J_{|\gamma|}(kr)$ term with $l = 0$ has to be absent in the expansion of the spatial wave function and Eq.(43) must be written as

$$\psi(r, \varphi) = \sum_{l=-\infty}^{\infty} N_l J_\nu(kr) e^{il\varphi} + J_{-|\gamma|}(kr) e^{i\pi\gamma/2}. \quad (45)$$

Here the summation is carried out with the omission of the $l = 0$ term, i.e. the $J_{|\gamma|}(kr)$ term. Now the scattering amplitude is given by

$$f(\varphi) = \sqrt{\frac{i}{2\pi k}} \frac{e^{i\varphi/2} \sin(\pi\gamma)}{\sin(\varphi/2)} + \sqrt{\frac{-i}{2\pi k}} \sin(\pi\gamma). \quad (46)$$

The second term on the right of this equation is absent when the electron spin lies in the z axis. The corresponding cross section is

$$\frac{d\sigma}{d\varphi} = \frac{\sin^2(\pi\gamma)}{2\pi k} \left(\frac{1}{\sin^2 \varphi/2} - 1 \right). \quad (47)$$

The cross section for the case when the spin of the scattered electron lies in the unit vector \mathbf{n} can be obtained by interpolating Eqs. (13) and (47), taking into account the conservation of spin projection on the z axis. In this case the second term, i.e., the “1” term, on the right of Eq. (47) must be replaced by the $1 - (\mathbf{n} \cdot \mathbf{n}_z)^2 = (\mathbf{n} \times \mathbf{n}_z)^2$ term and the cross section takes the form (20).

IV. FIRST BORN APPROXIMATION OF ELECTRON SCATTERING OFF A FLUX TUBE OF ZERO RADIUS

Let us briefly discuss the scattering of spin-polarized electrons in an AB potential in the first Born approximation. As was shown above the cross section does not depend on the radius of a flux tube so we consider the case when the magnetic field is restricted to a flux tube of zero radius.

If the effective delta potential is positive, the correction to the scattering amplitude which may arise from the inclusion of the additional potential is angle-independent and is very small compared with the usual AB term for small scattering angle φ . This term can be estimated if one considers the scattering of a free particle in the delta potential in the xy plane.

This part of scattering amplitude can be obtained by the formula [9]

$$f(\varphi) = \sqrt{\frac{k}{2i\pi}} \int \left(e^{2i\delta_{ph}(x)} - 1 \right) e^{-2ikx \sin(\varphi/2)} dx, \quad (48)$$

where

$$\delta_{ph}(x) = -\frac{\pi|e|Bs\delta(x)}{4\hbar kc} \quad (49)$$

is the phase shift. In the first Born approximation the scattering amplitude (48) can easily be obtained in the form

$$f(\varphi) = -\frac{\pi|e|Bs}{\sqrt{2ik\pi}\hbar c}. \quad (50)$$

One sees that the corresponding correction to the cross sections is the same for both the attractive and repulsive potential and for $s = \pm 1$. This formula coincides with the first term of the expansion on γ of second term on the right of Eq. (46).

We see that the scattering amplitude of particles in the two-dimensional delta potential does not depend on the scattering angle at any energy of particles. This result generalizes to the two-dimensional case the known exact result about the equality of amplitude for toward- and backward-scattering in an one-dimensional repulsive delta potential for any energy of particles (see, for instance, [11]).

V. POSSIBILITY OF BOUND ELECTRON STATE IN AN AHARONOV-BOHM POTENTIAL

In [8] it was shown that for an AB flux tube of zero radius a bound state of the electron may exist if the magnetic flux is suitably quantized. It was also shown that the occurrence of such bound state can modify the scattering states, but the total cross-section is unaffected.

Now we would like discuss whether a bound electron state may occur in the quantum system in which the flux tube has a finite radius. The wave function of such a bound state must belong to the negative energy spectrum, and corresponds to a probability density which is concentrated near the flux tube at the point $\mathbf{r} = 0$ and vanishes at large r . In the range $r > R$ there is only one solution of Eq. (7) with $E < 0$, which is expressed through the MacDonald function $K_0(z)$ of argument $z = r\sqrt{2m|E|/\hbar^2}$. This solution must be matched with the corresponding negative energy solution in the range $r < R$ at the point $r = R$.

In the range $r < R$ the solution, which could be matched with the MacDonald function $K_0(z)$, is the confluent hypergeometric function $\Psi(a, 1; x)$ with $l = 0$. It is well to note that the electron wave function can be normalized also only for $l = 0$. At small x , the function $\Psi(a, 1; x)$ becomes

$$\Psi(a, 1; x) \cong -\frac{1}{\Gamma(a)}(\ln x - \mathbf{C}), \quad (51)$$

where $\Gamma(a)$ is the gamma function of argument $a = 1 - E/\hbar\omega$ for $s = -1$ and $a = -E/\hbar\omega$ for $s = 1$, and $\mathbf{C} = 0.577\dots$ is the Euler constant. For negative E when the energy of bound state is very small $E \rightarrow -0$ it follows from Eq. (51) that $\Psi(a, 1; x)$ for $s = 1$ becomes $\Psi(a \rightarrow 1, 1; x) \rightarrow \ln x$ and $\Psi(a \rightarrow +0, 1; x)$ for $s = -1$ becomes $\Psi(a \rightarrow 0, 1; x) \rightarrow a \ln x$. But $\Psi(a, 1; x)$ behaves as

$$\Psi(a, 1; x) \cong -\frac{1}{(\Gamma(a))^2} e^x x^{a-1} \ln x \quad (52)$$

at large x and so for the necessary values of the parameter $a \rightarrow +0, 1$ the wave functions cannot be matched. Hence, we conclude that the bound electron state cannot exist if the AB flux has a finite radius.

VI. SUMMARY

In this paper we have considered the scattering of spin-polarized electrons in an AB vector potential. By solving the Pauli equation in 3+1 dimensions, taking into account explicitly the interaction between the three-dimensional spin magnetic moment of electron and magnetic field, we obtain expressions for the scattering amplitude and the cross section for spin-polarized electron scattered off a flux tube of small radius. We have also shown that bound electron states cannot occur in this quantum system. We have also computed the scattering amplitude of spin-polarized electron scattered off a flux tube of zero radius by the Born approximation, and have shown that it is consistent with the exact result.

Other than the AB phase, another interesting phase in quantum mechanics is the Aharonov–Casher phase. This is the phase acquired by an electron propagating in an electric field when the spin-orbit coupling due to the interaction of the moving magnetic moment with the electric field needs to be taken into account [13, 14]. We shall consider the Aharonov–Casher effect in the scattering of spin-polarized neutral fermions with anomalous magnetic moment elsewhere.

APPENDIX: SCATTERING OF SPIN-POLARIZED DIRAC ELECTRONS IN 2+1 DIMENSIONS

In this appendix we shall demonstrate that the results of [4, 6] for the scattering of relativistic spin-one-half electrons in 2 + 1 dimensions and the corresponding results presented here for the scattering of nonrelativistic spin-1/2 electrons in 3 + 1 dimensions coincide. For this we follow the approach of [6] by introducing the particle spin in the two-component Dirac equation as a new parameter. But unlike the treatment in [6] which assumes the magnetic field configuration in Eq. (1), we take the potential to be given by Eqs. (21) and (22) in the absence of a third partial (A_z) component. We shall adopt the units where $c = \hbar = 1$.

The Dirac equation for a particle of mass m and charge $e \equiv -e_0 < 0$ ($e_0 > 0$) in 2+1 dimensions in the absence of a third partial coordinate in the potential A_μ is

$$(\gamma^\mu \hat{P}_\mu - m)\Psi = 0. \quad (\text{A.1})$$

Here the Dirac γ^μ matrices are conveniently defined in terms of the Pauli spin matrices as (see, [6, 7])

$$\gamma^0 = \sigma_3, \quad \gamma^1 = is\sigma_1, \quad \gamma^2 = i\sigma_2, \quad (\text{A.2})$$

and s is a new parameter characterizing twice the spin value $s = \pm 1$ for spin “up” and “down”, respectively, $\hat{P}_\mu = -i\partial_\mu - eA_\mu$ is the generalized electron momentum operator. The corresponding Dirac Hamiltonian is

$$\mathcal{H} = \sigma_1 P_2 - s\sigma_2 P_1 + \sigma_3 m + eA_0. \quad (\text{A.3})$$

We seek solutions of Eq. (A.1) in the form [12]

$$\Psi(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \exp(-iEt + il\varphi) \psi_l(r, \varphi), \quad (\text{A.4})$$

where E is the electron energy, l is an integer, and $\psi_l(r, \varphi)$ is a two-component function (*i.e.* a 2-spinor)

$$\psi_l(r, \varphi) = \begin{pmatrix} f_l(r) \\ g_l(r)e^{is\varphi} \end{pmatrix}. \quad (\text{A.5})$$

Taking into account the easily checked relations

$$sP_1 \pm iP_2 = -ie^{\pm is\varphi} \left[s \frac{\partial}{\partial r} \pm \left(\frac{i}{r} \frac{\partial}{\partial \varphi} - \frac{e_0 B}{r} \right) \right], \quad (\text{A.6})$$

for the functions $f_l(r)$ and $g_l(r)$ we obtain the equations

$$\begin{aligned} s \frac{df_l}{dr} - \frac{l + e_0 B}{r} f_l + (E + m) g_l &= 0, \\ s \frac{dg_l}{dr} + \frac{s + l + e_0 B}{r} g_l - (E - m) f_l &= 0, \end{aligned} \quad (\text{A.7})$$

in the range $r > R$ and

$$\begin{aligned} s \frac{df_l}{dr} - \left(\frac{l}{r} + \frac{e_0 H r}{2} \right) f_l + (E + m) g_l &= 0, \\ s \frac{dg_l}{dr} + \left(\frac{l + s}{r} + \frac{e_0 H r}{2} \right) g_l - (E - m) f_l &= 0, \end{aligned} \quad (\text{A.8})$$

in the range $r > R$.

Eliminating, for instance $g_l(r)$, we obtain the differential equation for $f_l(r)$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + E^2 - m^2 - \frac{l^2}{r^2} - \frac{(e_0 H r)^2}{4} - e_0 H(l + s) \right) f_l(r) = 0, \quad (\text{A.9})$$

in the range $r < R$ and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + E^2 - m^2 - \frac{(l + e_0 B)^2}{r^2} \right) f_l(r) = 0 \quad (\text{A.10})$$

in the range $r > R$. It is seen that Eqs. (A.9) and (A.10) for a tube of small radius up to the trivial replacement $E^2 - m^2 = 2mE$ coincide respectively with the Eq. (24) for the range $r < R$ and Eq. (7) for the range $r > R$.

In the range $r > R$, the electron wave function has the form

$$\Psi_p(r, \varphi) = e^{-iEt + i\ell\varphi} \sqrt{\frac{\pi p}{2E}} \left(\pm \frac{\sqrt{E + m} J_\nu(pr)}{\sqrt{E - m} e^{is\varphi} J_{\nu \pm s}(pr)} \right). \quad (\text{A.11})$$

Here $p = \sqrt{E^2 - m^2}$, and $J_\nu(pr)$ is the Bessel function. In the range $r > R$ the linearly-independent solutions for $f_l(r)$ are

$$f_l(r) = a_l J_\nu(pr) + b_l J_{-\nu}(pr), \quad (\text{A.12})$$

where $J_\nu(pr)$ and $J_{-\nu}(pr)$ are the usual Bessel functions and

$$\nu = |l + \gamma| \neq 0, \quad \gamma = e_0 B > 0, \quad (\text{A.13})$$

a_l and b_l are constants. For $\nu > 0$ the regular solution is

$$f_l(r) = J_\nu(pr). \quad (\text{A.14})$$

For $\nu = 0$ the linearly-independent solutions for $f_l(r)$ are the Bessel ($J_0(pr)$) and Neumann ($N_0(pr)$) functions.

Assuming that the incident electron wave is moving from the left to the right along the x -axis. The upper component of the incident wave is $\psi = e^{ipx}$. The electron wave function in the potential (2) must have the asymptotic form

$$\psi_p(r, \varphi) = \left(\frac{1}{-ip/(E + m)} \right) e^{ipx + ie_0 B \varphi} + \left(\frac{1}{ip/(E + m)} \right) \frac{f(\varphi)}{\sqrt{r}} e^{ipr} \quad (\text{A.15})$$

as $r \rightarrow \infty$. Here $f(\varphi)$ is the scattering amplitude.

Writing $\psi(r, \varphi)$ in the form

$$\psi(r, \varphi) = \sum_{l=-\infty}^{\infty} A_l J_\nu(pr) e^{i\ell\varphi}, \quad (\text{A.16})$$

it is easy to show that

$$A_l = e^{-i(\pi/2)|l+e_0B|}. \quad (\text{A.17})$$

The scattering amplitude is proportional to $S_l - 1 \equiv e^{2i\delta_l} - 1$, where $\delta_l = (\nu - l)\pi = e_0B\pi \equiv e_0\Phi/2\hbar c$ are the partial phase shifts. They depend upon only the total magnetic flux Φ .

The coefficient before the term e^{ipr}/\sqrt{r} is the standard AB amplitude for the scattering of nonrelativistic particles

$$f_{\text{AB}}(\varphi) = \frac{1}{\sqrt{2\pi i}} \frac{e^{-i\varphi(n-1/2)} \sin(e_0\Phi/2)}{\sin(\varphi/2)}. \quad (\text{A.18})$$

Here $e_0\Phi = 2\pi n + 2\pi\Delta$ where n is an integer, and $-1/2 \leq \Delta \leq 1/2$. Amplitude (A.18) was first calculated in Ref. [4]. This part of the scattering amplitude is unaffected by the spin parameter s .

The effect of spin, leading to Eq. (20), can be considered in full analogy with Sect. III of this paper for the upper components.

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